BABINET'S PRINCIPLE FOR PLANE OBSTACLES

By

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Abstract

The electromagnetic form of Babinet's principle for a plane screen differs from the corresponding principle in optics.[1] It was first observed by Booker[2] and has been proved by Booker, Copson[3] and Meixner.[4] The proof given here makes use of vector field representations due to Levine and Schwinger[5] and has the advantage of compactness and simplicity.

The Vector Field Representations

A well-known theorem states that electromagnetic fields which satisfy Maxwell's equations throughout a region are uniquely determined by the values of the tangential components of the electric or magnetic vectors on the bounding surface of the region. Representations of the interior fields in terms of these boundary values have been derived by Levine and Schwinger[5] using dyadic Green's functions which are defined as follows

(a) The electric field dyadic Green's function $\Gamma^{(1)}(\vec{r}, \vec{r}')$ satisfies the inhomogeneous vector wave equation

$$\nabla \times \nabla \times \Gamma^{(1)}(\vec{r}, \vec{r}') - k^2 \Gamma^{(1)}(\vec{r}, \vec{r}') = \epsilon \delta(\vec{r} - \vec{r}') \tag{1}$$

and the boundary condition

$$\hat{n} \times \Gamma^{(1)}(\vec{r}, \vec{r}') = 0 \quad \vec{r} \text{ on } S. \tag{2}$$

Here ϵ represents the unit dyadic, $\hat{n}$ the outward normal at the surface S, and $\delta(\vec{r} - \vec{r}')$ is the delta-function defined by

$$\int_v dv \delta(\vec{r} - \vec{r}') = 1 \quad \vec{r}' \text{ in } v$$

$$\int_v dv \delta(\vec{r} - \vec{r}') = 0 \quad \vec{r}' \text{ not in } v \tag{3}$$

(b) The magnetic field dyadic Green's function $\Gamma^{(2)}(\vec{r}, \vec{r}')$ satisfies the inhomogeneous vector wave equation

$$\nabla \times \nabla \times \Gamma^{(2)}(\vec{r}, \vec{r}') - k^2 \Gamma^{(2)}(\vec{r}, \vec{r}') = \epsilon \, \delta(\vec{r} - \vec{r}') \tag{4}$$

and the boundary condition

$$n \times \nabla \times \Gamma^{(2)}(\vec{r}, \vec{r}') = 0 \qquad \vec{r} \text{ on } S. \tag{5}$$

These dyads, $\Gamma^{(1)}$, $\Gamma^{(2)}$, may be interpreted as operators which transform a unit electric or magnetic current vector $\hat{j}_E$, $\hat{j}_M$ at r' in a region bounded at S by perfectly conducting walls into an electric or magnetic field at $\vec{r}$ in the same region. That is

$$E(\vec{r}) = \frac{4\pi i k}{c} \Gamma^{(1)}(\vec{r}, \vec{r}') \cdot \hat{j}_E$$

and

$$H(\vec{r}) = \frac{4\pi i k}{c} \Gamma^{(2)}(\vec{r}, \vec{r}') \cdot \hat{j}_M.$$

(c) The free-space dyadic Green's function $\Gamma^{(0)}(\vec{r}, \vec{r}')$ satisfies a vector wave equation like (1) and the radiation condition at infinity. It may be interpreted as the operator in an infinite region which transforms unit currents into fields. Thus, in an unbounded region,

$$E(\vec{r}) = \frac{4\pi i k}{c} \Gamma^{(0)}(\vec{r}, \vec{r}') \cdot \hat{j}_E$$

and

$$H(\vec{r}) = \frac{4\pi i k}{c} \Gamma^{(0)}(\vec{r}, \vec{r}') \cdot \hat{j}_M.$$

The free-space dyadic Green's function has the closed form

$$\Gamma^{(0)}(\vec{r}, \vec{r}') = (\epsilon - \frac{i}{k^2} \nabla \nabla') \frac{\exp(ik|\vec{r} - \vec{r}'|)}{4\pi |\vec{r} - \vec{r}'|} = \Gamma^{(0)}(\vec{r}', \vec{r}). \tag{6}$$

In a bounded region, $\Gamma^{(1)}(\vec{r}, \vec{r}')$ and $\Gamma^{(2)}(\vec{r}, \vec{r}')$ will depend upon the geometric shape of the bounding surface S. However, all dyadic Green's functions have the symmetry properties:

$$\Gamma(\vec{r}, \vec{r}') = [\Gamma(\vec{r}', \vec{r})]^T$$
$$\nabla \times \Gamma^{(1)}(\vec{r}, \vec{r}') = [\nabla' \times \Gamma^{(2)}(\vec{r}', \vec{r})]^T \tag{7}$$

where Γ^T denotes the transposed dyadic of Γ.

The desired representations of the fields in a region in terms of the boundary values on its surface can now be found by making use of Green's second vector identity.

$$\int_S dS\,\hat{n} \cdot [\,B \times (\nabla \times A) - A \times (\nabla \times B)\,]$$

$$= \int_V dv\,[\,A \cdot \nabla \times (\nabla \times B) - B \cdot \nabla \times (\nabla \times A)\,] \qquad (8)$$

If the vector wave equations satisfied by the electromagnetic fields and the dyadic Green's functions are invoked, the appropriate choice of A and B leads to the following pairs of integral formulas:[5]

$$E(\vec{r}) = - \int_S dS'(\hat{n}' \times E(\vec{r}')) \cdot (\nabla' \times \Gamma^{(1)}(\vec{r}', \vec{r}))$$

$$H(\vec{r}) = ik \int_S dS'(\hat{n}' \times E(\vec{r}')) \cdot \Gamma^{(2)}(\vec{r}', \vec{r}) \qquad\qquad (9)$$

$$E(\vec{r}) = - ik \int_S dS'(\hat{n}' \times H(\vec{r}')) \cdot \Gamma^{(2)}(\vec{r}', \vec{r})$$

$$H(\vec{r}) = - \int_S dS'(\hat{n}' \times H(\vec{r}')) \cdot \nabla' \times \Gamma^{(1)}(\vec{r}', \vec{r}) \qquad\qquad (10)$$

$$E(\vec{r}) = - ik \int_S dS'(\hat{n}' \times H(\vec{r}')) \cdot \Gamma^{(0)}(\vec{r}', \vec{r})$$

$$H(\vec{r}) = - \int_S dS'(\hat{n}' \times H(\vec{r}')) \cdot \nabla' \times \Gamma^{(0)}(\vec{r}', \vec{r}) \qquad\qquad (11)$$

where $\hat{n}'$ is the outward normal from S. It should be noted that in (11) one part of S is the surface of an infinitely large sphere.

2. Integral Equation for the Aperture Field and
Screen Current for a Perforated Plane Conducting Screen

The integral formulas found in the previous section can be applied
to the diffraction of electromagnetic waves by a perforated plane conducting
screen. For convenience the screen is oriented in the plane $z = 0$, and the
half-spaces $z > 0$ and $z < 0$ are considered separately. The surfaces of
integration include the perfectly conducting screen S_g on which $\hat{z} \times \vec{E} = \hat{z} \cdot \vec{H} = 0$,
the aperture S_a, and the surface of a large hemisphere centered on the aperture.
If the dyadic Green's function for each half-space, $z \lessgtr 0$, is denoted by $\Gamma_\pm$, then
(9) can be reduced to

$$\left. \begin{array}{l} E(\vec{r}) = \displaystyle\int_{S_a} \hat{z} \times E(\vec{\rho}\,') \cdot \nabla' \times \Gamma_+^{(1)}(\vec{\rho}\,', \vec{r}) dS' \\ \\ H(\vec{r}) = -ik \displaystyle\int_{S_a} dS'(\hat{z} \times E(\vec{\rho}\,')) \cdot \Gamma_+^{(2)}(\vec{\rho}\,', \vec{r}) \end{array} \right\} z \geq 0 \qquad (12)$$

$$\left. \begin{array}{l} E(\vec{r}) = E^{inc} + E^{ref} - \displaystyle\int_{S_a} \hat{z} \times E(\vec{\rho}\,') \cdot \nabla' \times \Gamma_-^{(1)}(\vec{\rho}\,', \vec{r}) dS' \\ \\ H(\vec{r}) = H^{inc} + H^{ref} + ik \displaystyle\int_{S_a} dS'(\hat{z} \times E(\vec{\rho}\,')) \cdot \Gamma_-^{(2)}(\vec{\rho}\,', \vec{r}) \end{array} \right\} z \leq 0 \qquad (13)$$

Here use has been made of the boundary condition on the screen and of
the radiation condition at infinity. The dyadic Green's functions involved in
(12) and (13) are related to the free-space dyadic Green's function as follows[5]

$$\Gamma_+^{(1)}(\vec{r}, \vec{r}\,') = \Gamma^{(0)}(\vec{r}, \vec{r}\,') - \Gamma^{(0)}(\vec{r}, \vec{r}\,' - 2\hat{z}\hat{z} \cdot \vec{r}\,') \cdot (\epsilon - 2\hat{z}\hat{z}), \quad z, z' \geq 0$$

$$(14)$$

$$\Gamma_+^{(2)}(\vec{r}, \vec{r}\,') = \Gamma^{(0)}(\vec{r}, \vec{r}\,') + \Gamma^{(0)}(\vec{r}, \vec{r}\,' - 2\hat{z}\hat{z} \cdot \vec{r}\,') \cdot (\epsilon - 2\hat{z}\hat{z}), \quad z, z' \geq 0$$

and

$$\Gamma_-(\vec{r}, \vec{r}') = \Gamma_+ (\vec{r} - 2(\hat{z} \cdot \vec{r})\hat{z}, \ \vec{r}' - 2(\hat{z} \cdot \vec{r}') \ \hat{z}) \qquad z, z' \leq 0 \qquad (15)$$

When (14) is post-multiplied scalarly by a vector $\hat{z} \times \epsilon(\vec{r}')$, it can be seen that for $z' = 0$

$$\Gamma_{\pm}^{(2)}(\vec{r}, \vec{\rho}') \cdot \hat{z} \times E(\vec{\rho}') = 2\Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot \hat{z} \times E(\vec{\rho}') \qquad (16)$$

With the identity (16) and the symmetry properties it is now possible to rewrite (12) and (13) in the form

$$\left. \begin{aligned} E(\vec{r}) &= \nabla \times \int_{S_a} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot (2 \hat{z} \times E(\vec{\rho}') \, dS' \\[2em] H(\vec{r}) &= -ik \int_{S_a} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot (2 \hat{z} \times E(\vec{\rho}')) \, dS' \end{aligned} \right\} \quad z \geq 0 \qquad (17)$$

$$\left. \begin{aligned} E(\vec{r}) &= E^{inc} + E^{ref} - \nabla \times \int_{S_a} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot (2 \hat{z} \times E(\vec{\rho}')) dS' \\[2em] H(\vec{r}) &= H^{inc} + H^{ref} + ik \int_{S_a} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot (2 \hat{z} \times E(\vec{\rho}')) dS' \end{aligned} \right\} \quad z \leq 0 \qquad (18)$$

These formulas require knowledge of the tangential electric field in the aperture, which is not available in general. It is possible, however, to formulate the integral equations for this function by evaluating (17) in the aperture and using the boundary conditions ✝

$$\left. \begin{aligned} \hat{z} \times H(\vec{\rho}) &= \hat{z} \times H^{inc}(\vec{\rho}) \\[1em] \hat{z} \cdot E(\vec{\rho}) &= \hat{z} \cdot E^{inc}(\vec{\rho}) \end{aligned} \right\} \quad \vec{\rho} \text{ in } S_a \qquad (19)$$

Then

$$\hat{z} \cdot E^{inc}(\rho) = \hat{z} \cdot \nabla \times \int_{S_a} \Gamma^{(0)}(\vec{\rho}, \vec{\rho}') \cdot (2 \hat{z} \times E(\vec{\rho}')) dS' \Bigg| \qquad (20)$$

$$\left. \vphantom{\int} \right\} \vec{\rho} \text{ in } S_a$$

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✝These boundary conditions are proved in T. R. 163.

$$\hat{z} \times H^{inc}(\vec{\rho}') = - ik \hat{z} \times \int_{S_a} \Gamma^{(0)}(\vec{\rho}, \vec{\rho}') \cdot (2\hat{z} \times E(\vec{\rho}')) dS'$$

which are the desired integral equations.

An analogous procedure will yield a pair of integral equations for the current on the screen. In this case the perforated plane conducting screen is considered to be an obstacle embedded in free space, and formula (11) is used to describe the fields. With the radiation condition, the symmetry property (7), and the notation

$$K(\vec{\rho}) = \hat{z} \times (H_{-}(\vec{\rho}) - H_{+}(\vec{\rho})) \qquad \vec{\rho} \text{ in } S_s \qquad (21)$$

(the subscripts + and - here denote the limiting values of H as $z \to 0$ from the positive and negative side of the screen respectively), it is found that

$$E(\vec{r}) = E^{inc}(\vec{r}) - ik \int_{S_s} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot K(\vec{\rho}') dS'$$

$$\tag{22}$$

$$H(\vec{r}) = H^{inc}(\vec{r}) - \nabla \times \int_{S_s} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot K(\vec{\rho}') dS'$$

On the screen, $\hat{z} \times \vec{E}(\vec{\rho}) = \hat{z} \cdot \vec{H}(\vec{\rho}) = 0$, so that

$$\hat{z} \times E^{inc}(\vec{\rho}) = ik\hat{z} \times \int_{S_s} \Gamma^{(0)}(\vec{\rho}, \vec{\rho}') \cdot K(\vec{\rho}') dS'$$

$$\hat{z} \cdot H^{inc}(\vec{\rho}) = \hat{z} \cdot \nabla \times \int_{S_s} \Gamma^{(0)}(\vec{\rho}, \vec{\rho}') \cdot K(\vec{\rho}') dS'$$

$$\vec{\rho} \text{ in } S_s \qquad (23)$$

These are the required integral equations for K.

3. Babinet's Principle

With the integral equations found above it is possible to give a compact

proof of Babinet's principle as formulated for diffraction of electromagnetic waves by a perforated plane conducting screen. This principle may be stated as follows:

Let an infinitely extended plane be divided into two regions S_1 and S_2. Let E_1 and H_1 be the diffracted fields produced when an electromagnetic wave $E_1^{inc} = U$, $H_1^{inc} = V$ is incident upon such a plane for which S_1 consists of apertures and S_2 of perfect conductors. Let E_2 and H_2 be the diffracted fields produced when the complementary electromagnetic wave $E_2^{inc} = -V$, $H_2^{inc} = U$ is incident upon the complementary plane for which S_2 consists of apertures and S_1 of perfect conductors. Then Babinet's principle states that on the shadow side of the plane

$$E_1(\vec{r}) + H_2(\vec{r}) = U(\vec{r})$$

$$\text{(24)}$$

$$H_1(\vec{r}) - E_2(\vec{r}) = V(\vec{r}) \ ,$$

and on the illuminated side of the plane

$$E_1(\vec{r}) - H_2(\vec{r}) = U^{ref}(\vec{r}) \ ,$$

$$\text{(25)}$$

$$H_1(\vec{r}) + E_2(\vec{r}) = V^{ref}(\vec{r}) \ ,$$

where U^{ref} and V^{ref} are the specularly reflected waves that arise from E_1^{inc}, H_1^{inc}, when the entire plane is perfectly conducting.

The proof of the principle proceeds directly from the integral equations. With $S_1 = S_a$, $S_2 = S_s$, and the incident wave $E_1^{inc} = U$, $H_1^{inc} = V$ (20) becomes

$$\left. \begin{array}{c} \hat{z} \cdot U(\vec{\rho}) = \hat{z} \cdot \nabla \times \int_{S_1} \overleftrightarrow{\Gamma}^{(U)}(\vec{\rho}, \vec{\rho}\,') \cdot (2\hat{z} \times E_1(\vec{\rho}\,')) dS' \\ \\ \hat{z} \times V(\vec{\rho}) = -ik\, \hat{z} \times \int_{S_1} \overleftrightarrow{\Gamma}^{(0)}(\vec{\rho}, \vec{\rho}\,') \cdot (2z \times E_1(\vec{\rho}\,') dS' \end{array} \right\} \begin{array}{c} \vec{\rho} \text{ in } S_1 \\ \\ \text{(26)} \end{array}$$

For the complementary plane $S_2 = S_a$, $S_1 = S_s$, and the complementary incident field is $E_2^{inc} = -V$, $H_2^{inc} = U$. Equation (23) can be rewritten

$$\hat{z} \times V(\vec{p}) = -ik\,\hat{z}\times \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{p},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\}$$

$$\hat{z} \cdot U(\vec{p}) = \hat{z} \cdot \nabla \times \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{p},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\} \quad \vec{p} \text{ in } S_1 \qquad (27)$$

From equations (26) and (27) it is evident that

$$2\hat{z} \times E_1(\vec{p}') = K_2(\vec{p}') \qquad \vec{p}' \text{ in } S_1 \tag{28}$$

This relation between the tangential component of the electric field in the apertures and the surface current on the complementary conductors makes it possible to write integral formulas for the diffracted fields E_1, H_1 and E_2, H_2 in terms of K_2 alone. Thus when S_1 corresponds to the apertures (17) and (18) give

$$E_1(\vec{r}) = \nabla \times \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{r},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\}$$

$$H_1(\vec{r}) = -ik \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{r},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\} \quad z \geq 0 \qquad (29)$$

$$E_1(\vec{r}) = U(\vec{r}) + U^{ref}(\vec{r}) - \nabla \times \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{r},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\}$$

$$H_1(\vec{r}) = V(\vec{r}) + V^{ref}(\vec{r}) + ik \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{r},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\} \quad z \leq 0 \quad (30)$$

For the complementary problem equation (22) will be used, and

$$E_2(\vec{r}) = -V(\vec{r}) - ik \int_{S_1} \overline{\overline{\Gamma}}^{(0)}(\vec{r},\vec{p}') \cdot K_2(\vec{p}')dS' \Biggr\} \tag{31}$$

$$H_2(\vec{r}) = U(\vec{r}) - \nabla \times \int_{S_1} \Gamma^{(0)}(\vec{r}, \vec{\rho}') \cdot K_2(\vec{\rho}') dS' \qquad (31)$$

Babinet's principle follows at once from equations (29) (30) and (31).

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References

1. M. Born, Optik, Verlag, Julius Springer, Berlin, 1933, p. 154.

2. H. G. Booker, J. Instn. Elect. Engrs. Part III, A 93, 620-626 (1946).

3. E. T. Copson, "An Integral-Equation Method of Solving Plane Diffraction Problems," Proc. Roy. Soc., A 186, 100, (1946).

4. J. Meixner "Strenge Theorie der Beugung Elektromagnetischer Wellen an der Vollkommen Leitenden Kreisscheibe" Naturforschung 3A, 506 (1948).

5. H. Levine and J. Schwinger, "On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen," Communications on Pure and Applied Mathematics, Vol. 3, No. 4. 1950.